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Commensurability effects on the spectra of integrable systems

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Abstract. The eigenvalue spectra of two simple integrable systems—a three-dimensional anisotropic harmonic oscillator (AHO) and a particle in a cuboid—are studied as the aspect ratios are changed from rational to irrational values. For rational aspect ratios (commensurable case) there are large number-theoretic degeneracies which grow with energy. These degeneracies disappear when the aspect ratios are irrational (incommensurable case). However, for a given set of rational aspect ratios, commensurable behaviour ensues only at large energies. The number of distinct eigenvalues per unit energy interval, which is indicative of the degree of degeneracies, is seen to approach a constant asymptotically for rational aspect ratios. The asymptotic constant is determined for all sets of aspect ratios for the AHO and is estimated semi-analytically for several sets of aspect ratios for the cuboid. The crossover from incommensurable to commensurable behaviour is studied by following a sequence of rational aspect ratios which approaches an irrational limit. A scaling form the crossover is suggested and explored numerically. A clear indication of scaling is seen for the AHO, while the evidence is suggestive but not definitive for the cuboid. The distribution of energy level spacings in the commensurable regime is determined for the AHO and a sequence of cuboids.

1. Introduction

The eigenvalue spectra of the quantum mechanical counterparts of classically integrable systems have been studied extensively. It is quite well established that fluctuations in such spectra occur on two distinct energy scales (Berry 1983, 1987, Subrahmanyam and Barma 1990): an outer scale which describes the bunching of many different energy eigenvalues, and an inner energy scale which is set by the mean spacing between successive eigenvalues. On the outer scale, the structure in the density of states is well understood in terms of closed periodic classical orbits (Gutzwiller 1971, Balian and Bloch 1972, Berry and Tabor 1977). By contrast, spectral structure on the inner energy scale is not so handily described in such terms; the spectrum on this scale is very sensitive to small changes of parameters which cause large fluctuations in eigenvalue degeneracies.

Here we are concerned with the occurrence of large degeneracies of individual energy eigenvalues in two simple integrable systems—a three-dimensional anisotropic harmonic oscillator (AHO) and a particle in a cuboid. We follow the changes in the energy level spectrum as the aspect ratios (ratios of frequencies for the harmonic oscillator, ratios of sides for the cuboid) are varied from rational to irrational values. When the aspect ratios are rational (commensurable case) and the energy is large enough, there are large number-theoretic degeneracies which are not associated with any obvious symmetry. Such 'arithmetical' degeneracies have been studied in a number of systems (Itzykson and Luck 1986, Barma and Subrahmanyam 1991). These degeneracies disappear when the aspect ratios are irrational (incommensurable case).

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For systems with equal volumes, the total number of states in a given energy range is roughly the same whether or not there are degeneracies, but the number of energy levels (distinct eigenvalues) is sensitive to degeneracies. Accordingly, we examine the total number of levels $N_{\text{level}}(e)$ up to e, where e is a dimensionless, scaled energy. In the incommensurable case, the absence of large degeneracies implies that $N_{\text{level}}(e)$ is of the same order as the total number of states, and thus shows the following behaviour:

$$\begin{aligned} N_{\text{level}}(e) &\sim e^3 \qquad \text{incommensurable AHO} \\ &\sim e^{3/2} \qquad \text{incommensurable cuboid.} \end{aligned} \tag{1}$$

In the commensurable case, $N_{\text{level}}(e)$ turns out to vary linearly with e if e is large enough, for both AHO and cuboid, i.e. the function

$$f(e) \equiv \frac{N_{\text{level}}(e)}{e} \tag{2}$$

asymptotes to a constant $h(\boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \equiv (\alpha_x, \alpha_y, \alpha_z)$ determines the aspect ratios

$$f(e) \xrightarrow{e \to \infty} h(\boldsymbol{\alpha}) \qquad \text{commensurable AHO and cuboid.} \tag{3}$$

This saturation property of $f(e)$ implies that for large e, mean degeneracies, given by the ratio of the number of levels to the number of states in an energy interval, are of order e^2 for the AHO and of order $e^{1/2}$ for the cuboid.

In general, characteristic incommensurable or commensurable behaviour (equations (1) and (3)) does not necessarily set in over the entire range of e. For instance, if the aspect ratios are rational, but in some sense close to irrational, there is a crossover as function of e. Characteristic rational behaviour (3) sets in only for e much larger than a crossover energy $e^*(\boldsymbol{\alpha})$, while for $e < e^*(\boldsymbol{\alpha})$ the behaviour is similar to that in the incommensurable case (1). In this paper, we study examples of such incommensurable–commensurable crossover in the AHO and the cuboid by examining a sequence of commensurable sets $\boldsymbol{\alpha}_m$ which approach an incommensurable set $\boldsymbol{\alpha}$ as $m \to \infty$. For each set of aspect ratios $\boldsymbol{\alpha}_m$, we study the function $f(e)$ and its asymptotic value $h(\boldsymbol{\alpha}_m)$. We are able to find $h(\boldsymbol{\alpha}_m)$ for all m for the AHO. For the cuboid, we derive upper bounds on $h(\boldsymbol{\alpha}_m)$ for the first few members of the sequence $\boldsymbol{\alpha}_m$ and present numerical evidence that in fact the bounds are saturated.

We also address the question of whether a scaling description of the crossover is valid for asymptotically large e and m, i.e. whether we can write

$$f(e, h_m) \approx h_m Y\left(\frac{e}{h_m^{\Delta}}\right) \tag{4}$$

in the limit $m \to \infty$, $e \to \infty$, with $y \equiv e/h_m^{\Delta} \to$ held fixed. Here $h_m \equiv h(\boldsymbol{\alpha}_m)$, Y is the scaling function and Δ is the crossover exponent which depends on the system (AHO or cuboid). Note that (1)–(4) imply that $Y(y) \to 1$ as $y \to \infty$ and $Y(y) \sim y^{1/\Delta}$ as $y \to 0$. We study the crossover function Y numerically, for both the AHO and the cuboid. We find that while there is good evidence for asymptotic scaling for the AHO (section 2.1), the evidence is suggestive but not definitive for the cuboid (section 2.2).

2. Incommensurable–commensurable crossover and scaling

2.1. The three-dimensional anisotropic harmonic oscillator

The eigenvalues of a three-dimensional anisotropic harmonic oscillator (AHO) with frequencies of vibration in the ratios $\alpha_x:\alpha_y:\alpha_z$ are given by

$$\varepsilon_{\boldsymbol{n}} = \hbar\omega_0(\alpha_x n_x + \alpha_y n_y + \alpha_z n_z) \qquad n_{x,y,z} = 0, 1, 2 \ldots \tag{5}$$

where $\boldsymbol{n} = (n_x, n_y, n_z)$ labels the eigenstates, and the zero-point energy $\frac{1}{2}\hbar\omega_0(\alpha_x + \alpha_y + \alpha_z)$ has been subtracted off. In terms of the dimensionless, scaled energy defined as $e = (\varepsilon/\hbar\omega_0)(\alpha_x\alpha_y\alpha_z)^{-1/3}$, the number of states with scaled energy less than or equal to e remains roughly the same as the aspect ratios are changed, and varies as e^3.

In the commensurable case, it proves convenient to take α's to be integers with no common divisor. Degeneracies come about from the various ways of representing an integer as a linear form $L_{\boldsymbol{\alpha}} = \alpha_x n_x + \alpha_y n_y + \alpha_z n_z$. Now, the asymptotic probability that an integer is of the linear form $L_{\boldsymbol{\alpha}}$ is unity if two of the components of $\boldsymbol{\alpha}$ do not have a common divisor (see appendix A). This implies that the total number of levels $N_{\text{level}}(e)$ grows as e for large e, and (3) holds with

$$h(\boldsymbol{\alpha}) = (\alpha_x\alpha_y\alpha_z)^{1/3}. \tag{6}$$

With irrationally related aspect ratios it is clear that no two sets (n_x, n_y, n_z) can correspond to the same energy, i.e. no degeneracies, implying that the total number of levels is equal to the total number of states (see (1)).

To study the irrational–rational crossover along the lines discussed in section 1, we consider a sequence $\boldsymbol{\alpha}_m$ which leads to aspect ratios $1:\tau:\tau^2$ in the limit $m\to\infty$. Here $\tau = (\sqrt{5}+1)/2$ is the golden mean. We choose α_x, α_y, α_z to be β_m^2, $\beta_m\beta_{m+1}$, β_{m+1}^2 respectively where β_m is the mth member of the Fibonacci sequence ($\beta_1 = \beta_2 = 1$, $\beta_m = \beta_{m-1} + \beta_{m-2}$).

Figure 1(a) shows numerically generated $f(e)$ plotted against e^2 for a few sets $\boldsymbol{\alpha}_m$ for the AHO. Each curve shows a crossover from irrational behaviour at low energies $e < e^*(\boldsymbol{\alpha}_m)$ to rational behaviour at large energies $e > e^*(\boldsymbol{\alpha}_m)$. However, the crossover energy $e^*(\boldsymbol{\alpha}_m)$ shifts to larger values as the generation index m gets larger. Scaling would be valid if the asymptotic constant $h(\boldsymbol{\alpha})$ essentially determines the behaviour of the whole function $f(e)$ as given in (4). In order to test the scaling form (4), in figure 1(b) we have plotted $f_m(e)/h_m$ against the scaling variable $y^2 \equiv e^2/h_m$ (corresponding to $\Delta = \frac{1}{2}$ in (4)) for several generations m. The curves for different m seem to approach a limit, suggesting that in fact a scaling form describes the incommensurable–commensurable crossover in the AHO.

2.2. *Particle in a cuboid*

Consider a free particle in a finite cuboid with periodic boundary conditions, with sides in the ratio $\alpha_x:\alpha_y:\alpha_z$. The energy eigenvalues are given by

$$\varepsilon_{\boldsymbol{n}} = \frac{\hbar^2}{2m}\frac{1}{L^2}(\alpha_x n_x^2 + \alpha_y n_y^2 + \alpha_z n_z^2) \qquad n_{x,y,z} = 0, \pm1, \pm2 \ldots. \tag{7}$$

where $\boldsymbol{n}$ labels the eigenstates. With our definitions, the lengths of the edges of the cuboid are $2\pi L/\sqrt{\alpha_x}$ etc. We define a dimensionless, scaled energy as $e \equiv (2m\pi^2/\hbar^2)\varepsilon L^2(\alpha_x\alpha_y\alpha_z)^{-1/3}$ so that the total number of states remains roughly the same as we change the aspect ratios. When the aspect ratios are irrationally related, the only degeneracies are those coming from $\pm n_{x,y,z}$, in which case the total number of energy levels is of the same order as the total number of states, which in turn grows as $e^{3/2}$.

In the commensurable case, we choose α's to be integers. There is a possibility of large degeneracies, coming from different ways of representing an integer as the

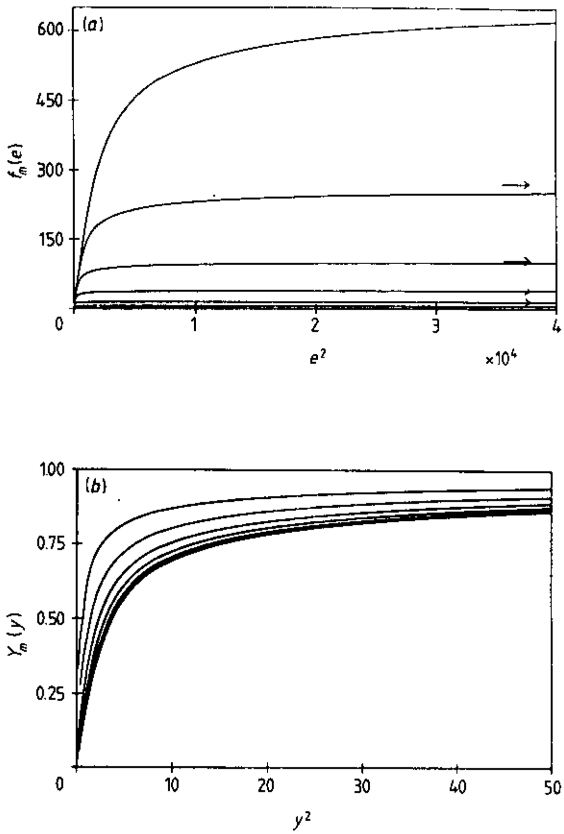

Figure 1. (a) The function $f_m(e)$ is plotted against e^2 for the generations $m = 3$ to 8 (moving upwards) for the AHO. Each of the curves displays incommensurable behaviour at small e and asymptotes, at large e, to a constant $h(\boldsymbol{\alpha}_m)$ marked by arrows. (b) $Y \equiv f(e)/h_m$ is plotted against the scaling variable $y^2 \equiv e^2/h_m$ for the AHO for generations $m = 3$ to 8 (moving downwards). The curves seem to approach a limiting curve as the generation index m increases, suggesting the existence of scaling for the incommensurable–commensurable crossover.

quadratic form $Q_\alpha = \alpha_x n_x^2 + \alpha_y n_y^2 + \alpha_z n_z^2$. Let $p(\boldsymbol{\alpha})$ denote the asymptotic probability that an integer is expressible in at least one way as the quadratic form Q_α. Then the asymptotic value of $f(e)$ is given by

$$h(\boldsymbol{\alpha}) = \frac{1}{\pi^2} p(\boldsymbol{\alpha})(\alpha_x \alpha_y \alpha_z)^{1/3}. \tag{8}$$

Finding $p(\boldsymbol{\alpha})$ for a general set $\boldsymbol{\alpha}$ is an unsolved number theoretical problem, but we have estimated it for some cases as explained further below.

In order to study incommensurable–commensurable effects, we consider the sequence $\boldsymbol{\alpha}_m$ with the three components chosen as β_m, β_{m+1}, β_{m+2} where β_m is the mth member of the Fibonacci sequence. As $m \to \infty$, the aspect ratios will approach the ratio $1:\tau:\tau^2$ as in section 2.1.

For the simplest case $\boldsymbol{\alpha} = (1, 1, 1)$ (corresponding to $m = 0$ of our sequence), Gauss proved (Landau 1927, Sierpinski 1964) that an integer k cannot be represented as

$n_x^2+n_y^2+n_z^2$ if and only if k belongs to the set $\mathscr{A}_0$

$$\mathscr{A}_0 \equiv \{4^l(8t+7);\ l, t=0, 1 \ldots\}. \tag{9}$$

That is, all integers which cannot be written as sums of three squares fall into patterns, each characterized by a period $4^l 8$ and a phase $4^l 7$. Since the patterns do not intersect, every member of $\mathscr{A}_0$ falls under one pattern only. The fraction of integers contained in $\mathscr{A}_0$ (i.e. the density of the set) is $\frac{1}{8}(1+\frac{1}{4}+\frac{1}{4^2}\ldots)=\frac{1}{6}$ which implies that $p(1,1,1)=\frac{5}{6}$. In a few simple cases other than $\boldsymbol{\alpha}=(1,1,1)$, Lebesgue, Dirichlet and Ramanujan provided the answers, following Gauss by finding patterns similar to (9) (Ramanujan 1927). But results for the cases we need were not known.

For each $\boldsymbol{\alpha}$ from the sequence under study, we found patterns in the set $\mathscr{A}_{\boldsymbol{\alpha}}$ of integers not representable as $Q_{\boldsymbol{\alpha}}$, and thus estimated the asymptotic probability $p(\boldsymbol{\alpha})$. However, it turns out that there are two important differences from the case $\boldsymbol{\alpha}=(1,1,1)$ discussed above. First, not all members of $\mathscr{A}_{\boldsymbol{\alpha}}$ belong to periodic patterns, though almost all do. Second, the patterns are not necessarily non-intersecting. Let $\mathscr{B}_{\boldsymbol{\alpha}}$ denote a subset of $\mathscr{A}_{\boldsymbol{\alpha}}$, comprising those members of $\mathscr{A}_{\boldsymbol{\alpha}}$ which are fully characterized by patterns. The sets $\mathscr{B}_{\boldsymbol{\alpha}}$ can be found analytically and the corresponding densities $1-p(\boldsymbol{\alpha})$ estimated. Results for several $\boldsymbol{\alpha}_m$ are displayed in table 1. Furthermore, by numerical enumeration of $\mathscr{A}_{\boldsymbol{\alpha}}$ we found that the density of $\mathscr{A}_{\boldsymbol{\alpha}}$ approaches that of $\mathscr{B}_{\boldsymbol{\alpha}}$, listed in table 1 for each $\boldsymbol{\alpha}$ (see figure 2(*a*)). This means that the leftovers of $\mathscr{A}_{\boldsymbol{\alpha}}$ (those members of $\mathscr{A}_{\boldsymbol{\alpha}}$ not characterized by patterns) constitute a vanishing fraction of integers, and do not contribute to $p(\boldsymbol{\alpha})$.

Table 1. The sets $\mathscr{B}_{\boldsymbol{\alpha}_m}$ of integers not of the quadratic form $Q_{\boldsymbol{\alpha}_m}$.

Generation index m	$\alpha_x, \alpha_y, \alpha_z$	The set $\mathscr{B}_{\boldsymbol{\alpha}}$ of integers not of the form $\alpha_x n_x^2+\alpha_y n_y^2+\alpha_z n_z^2$	Asymptotic fraction $1-p(\boldsymbol{\alpha}_m)$	$h(\boldsymbol{\alpha}_m)$
0	1, 1, 1	$4^l(8t+7)$	1/6	0.0844
1	1, 1, 2	$4^l(16t+10)$	1/12	0.1170
2	1, 2, 3	$4^l(16t+10)$	1/12	0.1687
3	2, 3, 5	$9^l(9t+6)$	1/8	0.2754
4	3, 5, 8	$9^l(9t+6)$; $(4t+2)$	11/32	0.3279
5	5, 8, 13	$(4t+3)$; $(8t+6)$; $4^l(64t+56)$	19/48	0.4922
6	8, 13, 21	$(4t+3)$; $(8t+6)$; $4^l(64t+56)$	19/48	0.7942
7	13, 21, 34	$9^l(9t+6)$; $4^l(16t+14)$; $49^l(49t+7, 14, 28)$	127/512	1.601
8	21, 34, 55	$9^l(9t+6)$; $4^l(16t+10)$; $49^l(49t+7, 14, 28)$	127/512	2.589
9	34, 55, 89	$(121t+22, 66, 77, 88, 110)$	5/121	5.343
10	55, 89, 144	$(4t+2)$; $(121t+22, 66, 77, 88, 110)$	136/484	6.483
11	89, 144, 233	$(4t+3)$; $(8t+6)$; $(9t+3, 6)$ $(16t+12)$; $4^l(128t+112)$	493/864	6.265
12	144, 233, 377	$(4t+3)$; $(8t+6)$; $(9t+3, 6)$; $(16t+12)$; $4^l(128t+112)$	493/864	10.14
13	233, 377, 610	$4^l(16t+14)$	1/12	35.01

We now turn to finding periodic patterns of the form $Pt+a$, and identifying $\mathscr{B}_{\boldsymbol{\alpha}}$, for a given quadratic form $Q_{\boldsymbol{\alpha}}$. The method of identifying a pattern for a given period P is outlined in appendix B. In principle, one has to try out all integers as candidates for periods. However, we observed that only prime numbers or their powers seem to figure as independent periods. Table 1 shows the patterns identified for several $\boldsymbol{\alpha}_m$,

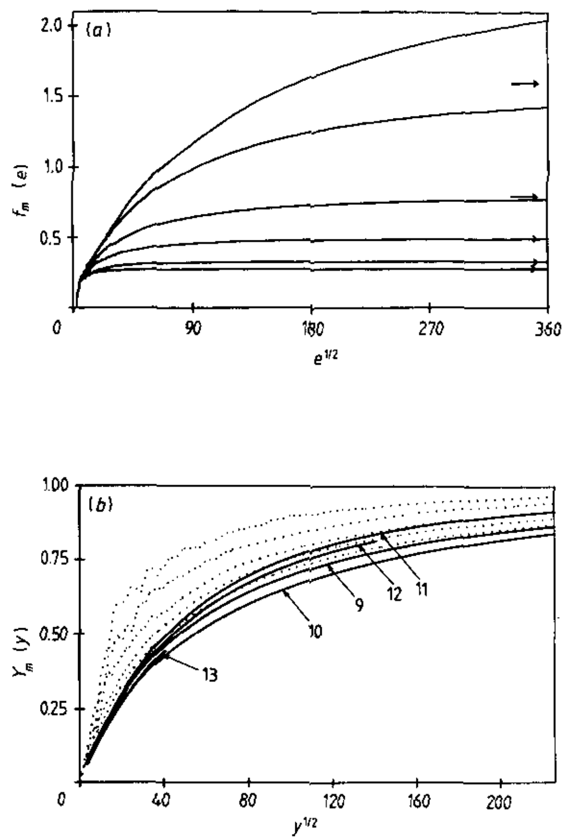

Figure 2. (*a*) The function $f_m(e)$ is plotted against $e^{1/2}$ for the cuboid for $m = 3$ to 8 (moving upwards). The asymptotic values $h(\boldsymbol{\alpha}_m)$ of $f(e)$ (given in table 1) are marked by arrows. (*b*) $Y \equiv f(e)/h_m$ is plotted against the scaling variable $y^{1/2} \equiv \sqrt{e}/h_m$ for the cuboid. The dotted curves are for $m = 4$ to 8 (moving downwards) while the solid curves for $m = 9$ to 13 are marked in the figure. Though there are fluctuations as the generation index m is changed, the solid curves are crowded in a relatively narrow region.

along with the corresponding densities. In each case, we have examined all integers up to 512 as candidate periods. We also tried all powers of primes up to 10 000 for primes below 100 in all cases. We believe our results for the densities of $\mathscr{B}_\alpha$ are accurate at least up to four significant figures. In principle, it is possible that we have not found the full set $\mathscr{B}_\alpha$ since we have not tried out all primes. However, our estimate of $p(\boldsymbol{\alpha})$ is a definite upper bound as it can be checked that any additional pattern not identified by us would increase the density of $\mathscr{B}_\alpha$ and thus decrease $p(\boldsymbol{\alpha})$. We tested our estimate of the density of $\mathscr{B}_\alpha$ against the density of numerically generated $\mathscr{A}_\alpha$. Due to computer limitations, we could examine integers up to 10^8 only for generating numerical data for $\mathscr{A}_\alpha$. For most cases given in table 1 (up to $m = 10$), we found good agreement, suggesting strongly that the bounds are saturated.

Using our estimate of $p(\boldsymbol{\alpha}_m)$ in (8) we find $h(\boldsymbol{\alpha}_m)$, which is the asymptotic value of $f(e)$ (see (3)). Figure 2(*a*) shows the numerically generated $f(e)$ plotted against $e^{1/2}$ for a few sets $\boldsymbol{\alpha}_m$. The corresponding values of $h(\boldsymbol{\alpha}_m)$ are indicated as well. The fact that each of the curves tends to asymptote (as $e \to \infty$) to the value given in table 1 implies that the leftovers (uncharacterized members of $\mathscr{A}_\alpha$) do not contribute to

$h(\boldsymbol{\alpha})$, and $f(e)$ exhibits commensurable behaviour asymptotically. Each curve displays the essential feature we concerned with, namely the incommensurable–commensurable crossover. Note that the incommensurable behaviour at low e is due to the leftovers of $\mathscr{A}_\alpha$ discussed above.

Turning to the scaling description of the incommensurable–commensurable crossover, we examine figure 2(*b*), which shows the scaling function $Y_m \equiv f(e)/h_m$ plotted against $y^{1/2} \equiv \sqrt{e}/h_m$ (corresponding to $\Delta = 2$ in (4)) for a few generations m. Though most curves crowd in patches (see figure 2(*b*)), the approach to a limit is not monotonic in m, in contrast to the situation for the AHO. To test whether the function $Y(y)$ in fact asymptotes to a limiting function as $m \to \infty$, in figure 3(*b*) we have plotted $Y(y^*)$ as a function of the generation index m for a few fixed values of the scaling variable $y = y^*$. Fluctuations in $Y(y)$ with changing m reflect the fluctuations of the number-theoretic probability $p(\boldsymbol{\alpha}_m)$ which is plotted as a function of m in figure 3(*a*). A comparison of figures 3(*a*) and 3(*b*) reveals that the fluctuations in $Y(y^*)$ are considerably smaller than those in $p(\boldsymbol{\alpha}_m)$; it is quite possible for $Y(y^*)$ to approach a limit as $e \to \infty$, $m \to \infty$, even though $p(\boldsymbol{\alpha}_m)$ may not tend to a limit as $m \to \infty$ (the answer to this

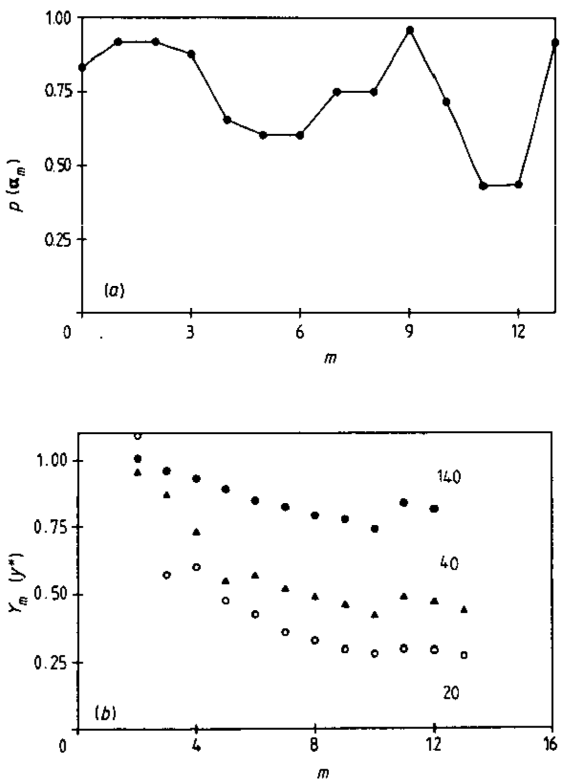

Figure 3. (*a*) The asymptotic probability $p(\boldsymbol{\alpha}_m)$ (given in table 1) is plotted against m for the cuboid. There are number-theoretic fluctuations as the generation index m is changed. (*b*) $Y_m(y^*)$ is plotted against m for three particular values of the scaling variable $y^* = 20$, 40, 140, for the cuboid. The amplitude of the fluctuations here is suppressed compared to fluctuations in $p(\boldsymbol{\alpha}_m)$, in (*a*), due to scaling, and $Y_m(y^*)$ seems to asymptote to a finite limit as $m \to \infty$.

question is still open). On the basis of the evidence presented (see figures 2(*b*) and 3(*b*)), we feel it is plausible that scaling obtains in a limiting sense.

3. Energy level statistics in the commensurable regime

We now turn to the question of distribution of energy levels (distinct eigenvalues). We are interested in the probability that two neighbouring levels have a spacing s. Since the average spacing between the energy levels varies differently with e in the incommensurable and commensurable regimes, and we would like a non-trivial asymptotic distribution for spacings, the choice of a proper 'energy' variable E is different in the two cases. E should be chosen such that the number of levels up to E grows linearly with E. Thus, $E = e$ in the commensurable regime for both the AHO and cuboid, whereas in the incommensurable regime $E = e^3$ for the AHO and $E = e^{3/2}$ for the cuboid. In terms of the energy variable E the average spacing between the levels is of order unity.

Let $P(s)\,\mathrm{d}s$ denote the probability that two successive energy levels have a spacing between s and $s+\mathrm{d}s$. In the incommensurable regime, $P(s)$ does not approach an asymptotic distribution for the AHO (Pandey and Ramaswamy 1991), while $P(s)$ is believed to approach a Poisson distribution for the cuboid (Shudo 1989). In the commensurable regime, as shown below, $P(s)$ can be calculated trivially for the AHO. For the cuboid, we can use the patterns $\mathscr{B}_{\alpha_m}$ (given in table 1), to determine $P_m(s)$. The answer depends on α_m, but for each m it is found that $P_m(s)$ consists of a finite number of delta functions, and is cut off beyond a finite value $s = s^*$.

3.1. *The* AHO

From (5), the energy level locations, in terms of $E = \varepsilon/\hbar\omega_0$, are specified by integers which can be represented as a linear form. For each generation m of the sequence we have considered in section 2, the fraction of integers that can be written as a linear form is unity as explained in appendix A. Thus the asymptotic spacing distribution for the energy levels is

$$P(1) = 1,\ P(s) = 0 \qquad \text{for } s > 1 \tag{10}$$

for all m.

3.2. *The cuboid*

The energy level locations are specified by integers which can be expressed as a quadratic form (see (7)). Let $\rho(s)$ be the density of isolated strings of s successive zeros i.e. integers which do not correspond to energy levels. The probability $P(s)$ is related to $\rho(s)$ (after normalization) by

$$P(s) = \frac{\rho(s-1)}{p(\boldsymbol{\alpha})} \qquad \text{for } s > 1. \tag{11}$$

$P(1)$ is determined from the normalization condition $\Sigma_1^\infty P(s) = 1$.

We now turn to finding $\rho(s)$. First consider the simple case of cube, i.e. $m = 0$ in table 1. We know that an integer is not a energy level if and only if it belongs to the set $\mathscr{A}_0$ given in (9). It is easy to see that three consecutive integers cannot belong to the above set $\mathscr{A}_0$, which implies that $\rho(s) = 0$ for $s > 2$. The argument proceeds by supposing the converse, i.e. $k, k+1, k+2 \in \{4^l(8t+7)\}$. Then there are two possibilities which we argue out separately.

(i) k is even. Since $8t+7$ is always odd, both k and $k+2$ should be of the form $4^l(8t+7)$ with $l \neq 0$. Then both k and $k+2$ would have to be divisible by 4 which is impossible.

(ii) k is odd. In this case both k and $k+2$ have to be of the form $8t+7$, which is again not possible.

Thus $\rho(s)=0$ for $s>2$. Also, we have the condition

$$\rho(1)+2\rho(2)=\tfrac{1}{6} \tag{12}$$

since the left-hand side should add up to the density of the set $\mathscr{A}_0$. Now $\rho(2)$ can be calculated by examining the number of pairs of integers $k, k+1$ belonging to the set $\mathscr{A}_0$. We need to find the solutions of

$$4^l(8t_1+7)=8t_2+7\pm 1. \tag{13}$$

The asymptotic fraction of integers satisfying the above equation is $1/4^l 8$, for $l>1$, which, on summing over l, leads to a probability 1/96 that a pair of integers belongs to $\mathscr{A}_0$. This implies that $\rho(2)=1/96$. From (12), it then follows that $\rho(1)=7/48$.

For cases other than $m=0$, since $\mathscr{A}_{\alpha_m}$ and $\mathscr{B}_{\alpha_m}$ lead to the same distribution, we can use our estimate of the sets $\mathscr{B}_{\alpha_m}$ (given in table 1) to find $\rho_m(s)$. The calculation follows the method used for $m=0$, though it is considerably involved in some cases; details are given elsewhere (Subrahmanyam 1991). The results are given in table 2. In each case the distribution is cut off, as in the case $m=0$.

Table 2. The densities $\rho(s)$ of isolated strings of s integers belonging to $\mathscr{B}_{\alpha_m}$.

Generation index m	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho(6)$	$\rho(7)$
0	7/48	1/96					
1, 2	1/12						
3	1/8						
4	7/32	1/16					
5, 6	6/48	5/48	1/48				
7	1025/6144	1247/36864	457/110592	31/110592	1/110592		
8	1379/8192	1721/55296	1115/221184	61/110592	1/36864		
9	5/121						
10	29/121	5/242					
11, 12	52/864	126/864	28/864	8/864	0	11/864	1/864
13	1/12						

4. Conclusions

We have studied commensurability effects on the eigenvalue spectra of a three-dimensional anisotropic oscillator and a particle in a cuboid. For large enough energies there are large number-theoretic eigenvalue degeneracies when the aspect ratios are rational, which disappear when the aspect ratios are irrational. The number of eigenvalues per unit energy interval $f(e)$ has been explored numerically for a sequence of rational aspect ratios which tends to an irrational limit. For a given rational set of aspect ratios, at low e, $f(e)$ exhibits incommensurable behaviour: $f(e) \sim e^2$ for the AHO and $f(e) \sim e^{1/2}$ for the cuboid. The commensurable behaviour, $f(e) \sim h(\boldsymbol{\alpha})$ for both the AHO and the cuboid, shows up at large e. The asymptotic value $h(\boldsymbol{\alpha})$ is determined for the AHO and for the first 14 members of the sequence of cuboids studied. An asymptotic scaling form for the incommensurable-commensurable crossover—according to which the asymptotic value $h(\boldsymbol{\alpha})$ should determine the whole function $f(e)$—is suggested and numerically explored. A clear indication is seen in favour of scaling for the AHO, whereas for the cuboid the evidence is suggestive.

In the commensurable regime, the nearest-neighbour energy level spacing distribution is shown to consist of a finite sequence of delta functions.

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Appendix A

In this appendix we find the asymptotic probability that an integer is of the linear form $L_{\boldsymbol{\alpha}} = \alpha_x n_x + \alpha_y n_y + \alpha_z n_z$.

Let $P(\alpha_x n_x + \alpha_y n_y + \alpha_z n_z)$ denote the asymptotic fraction of integers of the form $L_{\boldsymbol{\alpha}}$ for a particular integer set $\boldsymbol{\alpha}$. Exploiting the simplicity of the linear form we see that (for $\alpha_y < \alpha_z$)

$$P(\alpha_x n_x + \alpha_y n_y + \alpha_z n_z) = P(\alpha_x n_x + \bar{\alpha}_y n_y + \bar{\alpha}_z n_z) \tag{A1}$$

where $\bar{\alpha}_y = \alpha_y$ and $\bar{\alpha}_z = \alpha_z - \alpha_y$. Similarly proceeding further we can get a new linear form with

$$\bar{\alpha}_z = \alpha_z m_1 + \alpha_y m_2 . \tag{A2}$$

One can see from the above equation that, given any arbitrary α_y and, α_z, one can find an integer doublet (m_1, m_2) such that $\bar{\alpha}_z = 1$. This implies

$$P(\alpha_x n_x + \alpha_y n_y + \alpha_z n_z) = P(\alpha_x n_x + \bar{\alpha}_y n_y + n_z) = 1. \tag{A3}$$

Note that if the components of $\boldsymbol{\alpha}$ have a common divisor b, then the asymptotic probability is $1/b$.

Appendix B

Here we show how to find a pattern $Pt + a$ amongst the integers not representable as $\alpha_x n_x^2 + \alpha_y n_y^2 + \alpha_z n_z^2$. An integer k can be written as $Pt + R_P(k)$, where t is any integer and $R_P(k)$ can take values $0, 1, \ldots, P-1$. That is $R_P(k)$ is just the remainder of k when divided by P. This defines a congruence modulo P, $k \equiv R_P(k)$ (mod P). Note that a fraction $1/P$ of integers are congruent to a(mod P), $a \in (0, 1 \ldots P-1)$.

Let us examine the possible remainders $R_P(k)$ (modulo P) that can be generated by the integers of the form $k=\alpha_x n_x^2+\alpha_y n_y^2+\alpha_z n_z^2$, for a given $\boldsymbol{\alpha}$:

$$R_P(k)=R_P[R_P(\alpha_x n_x^2)+R_P(\alpha_x n_x^2)+R_P(\alpha_x n_x^2)]. \quad \textbf{(B1)}$$

Now each one of the three terms in the parentheses on the right-hand side is a number from $(0, 1, \ldots, P-1)$, but it is not necessary that it takes on every value in the interval (in fact the number of distinct values for each one of the squares in (B1) is upper bounded by $(P+1)/2$). We can check what values $R_P(k)$ can take (modulo P) by adding the three integers in the parentheses (of (B1)). If it is found that a particular value $a_i \in (0, 1, \ldots, P-1)$ cannot be generated by the above procedure, it immediately implies that any integer $k=Pt+a_i$ cannot be represented as $\alpha_x n_x^2+\alpha_y n_y^2+\alpha_z n_z^2$.

In order to check the allowed values of $R_P(\alpha_x n_x^2)$, it suffices to vary n_x between 0 and $P-1$. Thus a finite calculation gives information about all the integers of the form $Pt+a$ (which constitute $1/P$ fraction of all the integers). This is the essential point of the above procedure. For a short period P, the whole scheme can be carried out by hand while for longer periods $P>100$ the scheme can be implemented on a computer.

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